The Bivariate Normal Distribution
The Two Distributions Are Uncorrelated

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The bivariate normal distribution is used to determine the **joint probability** of pulling a random variate with a given value from one normal distribution and pulling another random variate with a given value from another normal distribution. To have a bivariate normal distribution or a joint normal distribution, both random variables need to be normally distributed and independent.

In this white paper we will develop the mathematics for the bivariate normal distribution and then use those equations to calculate joint probabilities for a hypothetical case where the distribution of random variable x and random variable y are uncorrelated.

Our Hypothetical Problem

The table below summarizes the parameters to our problem...

μ_x	=	The mean of the random variate x	=	6.00
σ_x	=	The standard deviation of the random variate x	=	5.00
μ_y	=	The mean of the random variate y	=	4.00
σ_y	=	The standard deviation of the random variate y	=	3.00

The problem: The random variate a is pulled from the distribution of x and the random variate b is pulled from the distribution of y. Using the table above, what is the joint probability that the random variate pulled from the distribution of x is less than 4.5 **and** the random variate pulled from the distribution of y is less than 3.5?

Building Our Model

The equation for the joint probability that x is less than a and y is less than b when the distributions of random variable x and random variable y are uncorrelated is...

$$\text{Prob}\Big[x \le a, y \le b\Big] = \text{Prob}\Big[x \le a\Big] \times \text{Prob}\Big[y \le b\Big] \tag{1}$$

We will define the function $f(x)$ to be the probability density function of the normal distribution of the random variable x and the function $g(y)$ to be the probability density function of the normal distribution of the random variable y. Using these definitions, we can rewrite Equation (1) above as...

$$\text{Prob}\Big[x \le a, y \le b\Big] = \int\limits_{-\infty}^{a} \int\limits_{-\infty}^{b} f(x)\, g(y)\, \delta y\, \delta x \tag{2}$$

We will define the variables μ_x and σ_x to be the mean and standard deviation, respectively, of the distribuion of the random variable x. The equation for the probability density function $f(x)$ is...

$$f(x) = \sqrt{\frac{1}{2\,\pi\,\sigma_x^2}}\,\text{Exp}\left\{-\frac{1}{2}\frac{(x-\mu_x)^2}{\sigma_x^2}\right\} \tag{3}$$

We will define the variables μ_y and σ_y to be the mean and standard deviation, respectively, of the distribuion of the random variable y. The equation for the probability density function $g(y)$ is...

$$g(y) = \sqrt{\frac{1}{2\,\pi\,\sigma_y^2}}\,\mathrm{Exp}\left\{-\frac{1}{2}\,\frac{(y-\mu_y)^2}{\sigma_y^2}\right\} \tag{4}$$

Using Equations (3) and (4) above, we can rewrite Equation (2) above as...

$$\begin{aligned}\mathrm{Prob}\Bigg[x \le a, y \le b\Bigg] &= \int\limits_{-\infty}^{a}\int\limits_{-\infty}^{b}\sqrt{\frac{1}{2\,\pi\,\sigma_x^2}}\,\mathrm{Exp}\left\{-\frac{1}{2}\,\frac{(x-\mu_x)^2}{\sigma_x^2}\right\}\sqrt{\frac{1}{2\,\pi\,\sigma_y^2}}\,\mathrm{Exp}\left\{-\frac{1}{2}\,\frac{(y-\mu_y)^2}{\sigma_y^2}\right\}\delta y\,\delta x\\ &= \int\limits_{-\infty}^{a}\int\limits_{-\infty}^{b}\frac{1}{2\,\pi\,\sigma_x\sigma_y}\,\mathrm{Exp}\left\{-\frac{1}{2}\left[\frac{(x-\mu_x)^2}{\sigma_x^2}+\frac{(y-\mu_y)^2}{\sigma_y^2}\right]\right\}\delta y\,\delta x \end{aligned} \tag{5}$$

We will define the function CNDF to be the cumulative normal distribution function. Using Equations (3) and (4) above, the equations for the cumulative normal distribution functions for the random variables x and y are...

$$\mathrm{CNDF[x](a)} = \int\limits_{-\infty}^{a} f(x)\,\delta x \ \text{ ...and... } \ \mathrm{CNDF[y](b)} = \int\limits_{-\infty}^{b} g(y)\,\delta y \tag{6}$$

Given that the distributions of x and y are not correlated, using Equation (6) above, the solution to Equation (5) above is...

$$\mathrm{Prob}\Bigg[x \le a, y \le b\Bigg] = \mathrm{CNDF[x](a)} \times \mathrm{CNDF[y](b)} \tag{7}$$

Note that the Excel equivalent of the cumulative normal distribution function in Equations (6) and (7) above is...

$$\text{CNDF(value)} = \text{NORM.DIST(value, mean, standard deviation, true)} \tag{8}$$

The Answer To Our Hypothetical Problem

Using Equation (7) above and the parameters to our problem, the solution to our problem in equation form is...

$$\mathrm{Prob}\Bigg[x \le 4.5, y \le 3.5\Bigg] = \mathrm{CNDF[x](4.5)} \times \mathrm{CNDF[y](3.5)} \tag{9}$$

Using Equation (8) above and the parameters to our problem, the values of the cumulative normal distribution functions are...

$$\begin{aligned}\text{CNDF[x](4.5)} = \text{NORM.DIST(4.5, 6, 5, true)} = 0.3821\\ \text{CNDF[y](3.5)} = \text{NORM.DIST(3.5, 4, 3, true)} = 0.4338\end{aligned} \tag{10}$$

Using Equations (9) and (10) above, the solution to our problem is...

$$\mathrm{Prob}\Bigg[x \le 4.5, y \le 3.5\Bigg] = 0.3821 \times 0.4338 = 0.1658 \tag{11}$$